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EFFECTS OF THE END FIXATION OF AIRPLANE STRUTS

By Alfred Teichmann

From Zeitschrift für Flugtechnik und Motorluftschiffahrt  
May 28, 1930

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS.

TECHNICAL MEMORANDUM NO. 582.

EFFECTS OF THE END FIXATION OF AIRPLANE STRUTS.\*

By Alfred Teichmann.

Various inquiries of the D.V.L. indicate that there is still considerable uncertainty concerning the problem of the effects of the end fixation of airplane struts as hitherto treated. In the present communication this problem will be discussed in as intelligible a manner as possible, with reference to the literature on the subject. This communication is in response to numerous requests. It contains no new information, and its method of presentation is not directly related to any of the works referred to.

Calculation of Struts in Frameworks with Rigid Joints

1. General Remarks on the Effects of Fixation

On the assumption that the deformations are small, the displacements of the joints of a framework can be represented by a linear system of equations. In certain loading conditions (buckling conditions), the denominator determinant of this system of equations disappears. Consequently, indeterminate and infinitely great joint displacements are produced, -i.e., the system becomes unstable and collapses.

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\*"Einspannwirkung bei Knickstäben in Flugzeug-Fachwerken." From Zeitschrift für Flugtechnik und Motorluftschiffahrt, May 28, 1930, pp. 249-254.

Contrary to the customary method of expression, not simply one member buckles, but the whole system acquires a buckling condition under a critical load. This occurs oftenest in frameworks with rigid joints, only when the stresses in a few members exceed the load, at which buckling would occur in nonrigid frictionless end supports - "natural" buckling load (Bleich).\*

The possibility that individual members may sometimes buckle is offset, however, by the greater clearness. The fact that the "natural" compression load can be exceeded in the compression members, is designated, in this method of presentation, as the "end-fixation effect." It is fundamentally wrong, in complex frameworks, to calculate with "end-fixation factors" which are independent of the structure and stressing of the whole system. If, for example, the "natural" buckling load is reached simultaneously in all compression members, no "end-fixation effect" is produced, except in special systems.

For simplicity in the determination of the buckling conditions, it is assumed that the compression members of the framework (before buckling begins) are subjected to only normal stresses. In reality, however, due to rigid joints, every member is subjected to bending moments resulting from deformations of the framework. The members are then subjected to ten-

\*An axially compressed lattice member, for example, may buckle before the natural buckling load is reached in a single member. In the first case, the buckling of the framework is comparable with so-called local buckling.

sile or buckling stresses.

Due to the resulting secondary stresses, failure occurs even below the theoretical buckling load, namely, when the failing stress of the critical member is exceeded. It is essential that these secondary stresses attain their maximum values in the annealed welded zones of the members, provided the case is not one of strongly bent compression members where, due to the added moment of the normal force, the maximum value of the secondary stresses may reside in the middle of the member.

Since these detrimental secondary stresses are ordinarily disregarded by assuming the members to have pin-end supports, it would be logical to disregard likewise the favorable effect of the end fixation. In any event, the consideration of the end-fixation effect must be omitted from the outset in the case of those members in which there is a possibility of strong secondary stresses.

## 2. Literature on the Calculation of the Buckling of Struts

For immediate practical use, a discussion of the theory of the buckling of a system of members in one plane is found in the book by F. Bleich (Reference 1, page 24). Along with the general theory, this book contains numerous simple approximation formulas. These problems are discussed separately by Bleich (References 2 and 3).

Another presentation of the theory of the buckling of a

planar system of members from its plane is contained in Zimmermann's book (Reference 4). Zimmermann's presentation is characterized by the introduction of the obvious conception of the fixation coefficient

$$m = \frac{\text{bending moment at end of member}}{\text{corresponding rotation of end cross section}},$$

that is, a measure for the end-fixation effect which a loaded or nonloaded strut can produce, or which it requires to prevent buckling. The book is a summary of Zimmermann's earlier works, which appeared in the reports of the Prussian Academy of Sciences (Preussische Akademie der Wissenschaften) in the years 1905 to 1925).

The stability of planar frameworks (including those with hinged joints) is treated in a general way in the works of Von Mises (Reference 5); Von Mises and Ratzersdorfer (References 6 and 7). The stability of space frameworks with any kind of joints is treated in the report of F. Bleich and H. Bleich (Reference 8). A future report by the same writers is announced. Other references are given in the above-mentioned works.

### 3. Estimation of the End-Fixation Effect

The accurate determination of the buckling load and of the buckling-bending stresses would require tedious calculations.

Consequently one must ordinarily be satisfied with estimates of the end-fixation effect.

The simplest way to estimate the buckling condition of a framework is to assume hinges in every joint. The buckling state is then given, when the "natural" buckling load in the most highly stressed compression member is determined.

Another and better approximation is obtained by combining in separate groups certain of the rigidly joined members and then joining these together into a single system by joints assumed to be perfectly flexible. The buckling condition of the whole system is then given when the buckling condition of the most highly stressed group is found (References 1 and 2).

In the calculation of a web member of a framework against buckling in the girder plane, this group includes both the immediately adjacent lower-flange members, which are in tension, and also, in so far as they are not fully utilized in compression, both the upper-flange members adjoining the web members (See also "Concluding Remarks," page 22). In the calculation of vertical members against buckling from the girder plane, the adjacent cross beams and braces are included.

It is obviously necessary to form the groups in the simplest way by combining every two consecutive flange members, or every three members meeting in one joint at any angle.

## 4. Buckling Formulas for Simple Groups of Members

## a) Groups of two successive members

α) Let the points 1, 0, and 2 be fixed in space, both members having the same direction (Fig. 1). In buckling "0" rotates. Consequently

$M_O^{1 \cdot 0}$  acts on 1.0 at "0"

$M_O^{2 \cdot 0}$  " " 2.0 " "0"

The cross section "0" rotates  $M_O^{1 \cdot 0} e^{1 \cdot 0}$  at 1.0

" " " "0" "  $M_O^{2 \cdot 0} e^{2 \cdot 0}$  " 2.0

$e$  = "unit rotation" of a strut end produced by a moment  $M = 1$ .  $\frac{1}{e} = m$  = fixation coefficient according to Zimmermann. For equilibrium

$$M_O^{1 \cdot 0} + M_O^{2 \cdot 0} = 0 \quad (1)$$

For continuity

$$M_O^{1 \cdot 0} e^{1 \cdot 0} - M_O^{2 \cdot 0} e^{2 \cdot 0} = 0 \quad (2)$$

This system of equations then allows finite values  $M_O^{1 \cdot 0}$ ,  $M_O^{2 \cdot 0}$  and hence finite rotations, if

$$e^{1 \cdot 0} + e^{2 \cdot 0} = 0$$

or

$$\frac{1}{e^{1 \cdot 0}} + \frac{1}{e^{2 \cdot 0}} = 0 \quad (\text{Buckling condition}).$$

The "unit rotation"  $e$  of the end cross section of a strut with free joints at both ends, resulting from a moment  $M = 1$ , is in the direction of this moment (Reference 9)

$$e = \frac{s}{E J} \frac{1}{\alpha^2} \left( 1 - \frac{\alpha}{\tan \alpha} \right)$$

whereby  $\alpha = s \sqrt{\frac{S}{E J}}$ ; +  $S$  = compression

or  $e = \frac{s}{E J} \frac{1}{3}$  provided  $S = 0$

or  $e = \frac{s}{E J} \frac{1}{\alpha^2} \left[ \frac{\alpha}{\tan \alpha} - 1 \right]$

whereby  $\alpha = s \sqrt{\frac{S}{E J}}$ ; +  $S$  = tension.

Detailed tables and charts for the evaluation of these expressions are found under various headings:  $c(\varphi)$ ,  $v'$ ,  $t$ , etc., in the works of Bleich (References 1 and 2); H. Muller-Breslau (Reference 9); Zimmermann (Reference 4) (Fig. 2).

If the dimensions and stresses, that is,  $\alpha$  and  $1/e$  of one strut is known, the buckling condition yields the value  $1/e$  of the other strut at which the system would buckle. If the value  $1/e$  of the last strut (taking note of the sign) is greater, the dimensions suffice. Ratzersdorfer gives special charts for the dimensioning of the two struts in his work (Reference 10).

## E x a m p l e 1

	Strut 1.0	Strut 2.0
s (cm)	80	60
J (cm <sup>4</sup> )	0.1513	0.0867
F (cm <sup>2</sup> )	0.406	0.408
S (kg) (load)	500	400
E (kg/cm <sup>2</sup> )	$2 \times 10^6$	$2 \times 10^6$
Natural buckling load (kg)	465	475
$\alpha = s \sqrt{\frac{S}{E J}}$	3.26	2.88
$\frac{1}{\alpha^2} \left(1 - \frac{\alpha}{\tan \alpha}\right)$ (Fig. 2)	-2.5	+1.5

"Unit rotation"  $e \left( \frac{1}{\text{kg/cm}} \right) = 6.6 \times 10^{-4} + 5.2 \times 10^{-4}$

Buckling condition  $e^{1.0} + e^{2.0} = 0$ .

For  $e^{1.0} = -6.6 \times 10^{-4}$  it is fulfilled, provided

$$e^{2.0} = +6.6 \times 10^{-4}.$$

The value  $1/e^{2.0}$  is therefore greater than  $1/e^{1.0}$ , and the assembly 1, 0, 2 is therefore safe against buckling. For the forces to be absorbed, the bar stresses are

$$1230 \text{ and } 980 \text{ kg/cm}^2.$$

Since these are below the elastic limit, the use of the value

$E = 2 \times 10^6 \text{ kg/cm}^2$  is justified.

β) The joint "0" is fixed in only one plane, as, for example, the middle joint of a K-strut, both members lying in the same direction (Fig. 3). Then the system "1, 0, 2" buckles throughout its whole length. In buckling "0" is shifted about  $y$  toward "0'" and rotates. Consequently

$M_O^{1\cdot0}$  acts on 1.0 at "0"

$M_O^{2\cdot0}$  " " 2.0 " "0"

The cross section "0" rotates from 1.0 toward the chord 1.0' by  $M_O^{1\cdot0} \times e^{1\cdot0}$  and from 2.0 toward the chord 2.0' by  $M_O^{2\cdot0} \times e^{2\cdot0}$ . The slope of the chord:

1.0' toward 1.0 is  $\vartheta_1 = y/s_1$

2.0' " 2.0 "  $\vartheta_2 = -y/s_2$ .

For equilibrium

$$M_O^{1\cdot0} + S_1 y + (S_2 - S_1) \frac{y}{s} s_1 = 0 \quad (1)$$

$$M_O^{2\cdot0} - S_2 y + (S_2 - S_1) \frac{y}{s} s_2 = 0 \quad (2)$$

For continuity

$$M_O^{1\cdot0} e^{1\cdot0} + \frac{y}{s_1} = M_O^{2\cdot0} e^{2\cdot0} - \frac{y}{s_2} \quad (3)$$

These equations yield finite values of  $M_O^{1\cdot0}$ ,  $M_O^{2\cdot0}$ ,  $y$  and hence also of  $\tau$ , when their denominator determinant

$$\begin{vmatrix} 1 & 0 & \frac{1}{s} (S_1 s_2 + S_2 s_1) \\ 0 & 1 & -\frac{1}{s} (S_1 s_2 + S_2 s_1) \\ e^{1\cdot0} - e^{2\cdot0} & & \frac{s}{s_1 s_2} \end{vmatrix} = 0$$

That is,

$$e^{1 \cdot 0} + e^{2 \cdot 0} = \frac{s^2}{s_1 s_2 (S_1 s_2 + S_2 s_1)} \text{ "buckling condition."}$$

If  $s_1 = s_2$ , as is usually the case in K struts, the buckling condition reads

$$e^{1 \cdot 0} + e^{2 \cdot 0} = \frac{8}{s (S_1 + S_2)} .$$

b) Group of three spatially related members. If the struts  $s_1$  and  $s_2$ , with junction points fixed in space, do not lie in the same direction, then the buckling load of the group  $s_1/s_2$  equals the "natural" buckling load of the greater-stressed one of the two struts, provided perfect torsion-free joints are assumed at the end supports. With this assumption, a third strut, spatially related to  $s_1/s_2$ , must be rigidly attached at the joint O, if any end-fixation effect is to follow. (As regards the effect of torsionally rigid end supports of the system  $s_1/s_2$ , see Reference 8.

If the system  $s_1/s_2$  is prevented in any way from buckling, then, as is obvious from Section a,  $\alpha$  (page 6), the condition for buckling from its plane is the same as before, if  $s_1$  and  $s_2$  lie in the same direction.

The buckling of a group of three spatially related struts, firmly united in all directions at the joint "O" and freely connected with the rest of the framework (regarded as rigid), is calculated as follows (Fig. 4): In buckling, the common joint "O" rotates through the angle  $t$ , thereby producing the

bending moment  $\underline{M}_O^{i,o}$  on the strut  $i,o$  at "O." This moment corresponds (on the assumption of a circular or annular cross section\* of the strut) to a rotation of the end cross section by  $\underline{M}_O^{i,o} e^{i,o}$  (in the direction of  $\underline{M}_O^{i,o}$ ).

A further rotation of the end cross section can be produced by the strut turning about its longitudinal axis, due to the torsional yielding of the end cross section. It is

$$c_i \underline{s}^{i,o}$$

( $\underline{s}^{i,o}$  = unit vector in the direction  $i,o$ .  $c_i$  = absolute torque.) For equilibrium

$$\sum_1^3 \underline{M}_O^{i,o} = 0 \quad (1)$$

For continuity

$$\underline{M}_O^{i,o} e^{i,o} + c_i \underline{s}^{i,o} = t \quad (i = 1,2,3) \quad (2)$$

Since no torsional moments can act in the struts, the moment vectors  $\underline{M}_O^{i,o}$  must be perpendicular to the axes of the struts.

$$\underline{M}_O^{i,o} \underline{s}^{i,o} = 0 \quad (3)$$

The equations resulting from the above groups, for the three components  $t_x$ ,  $t_y$  and  $t_z$ , yield finite values, if the denominator determinants are

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\*Or any other cross section for which the two principal inertia moments are equal.

$$0 = \begin{vmatrix} \sum_1^3 \frac{1}{e^{1,0}} [(x_i')^2 - 1] & \sum_1^3 \frac{x_i' y_i'}{e^{1,0}} & \sum_1^3 \frac{x_i' z_i'}{e^{1,0}} \\ \sum_1^3 \frac{y_i' x_i'}{e^{1,0}} & \sum_1^3 \frac{1}{e^{1,0}} [(y_i')^2 - 1] & \sum_1^3 \frac{y_i' z_i'}{e^{1,0}} \\ \sum_1^3 \frac{z_i' x_i'}{e^{1,0}} & \sum_1^3 \frac{z_i' y_i'}{e^{1,0}} & \sum_1^3 \frac{1}{e^{1,0}} [(z_i')^2 - 1] \end{vmatrix}$$

in which  $x_i$ ,  $y_i$  and  $z_i$  are the  $x$ ,  $y$ ,  $z$  coordinates of the point  $i$  ( $i = 1, 2, 3$ ) with respect to a system of coordinates with the origin at  $O$  and

$$x' = \frac{x}{s}, \quad y' = \frac{y}{s} \quad \text{and} \quad z' = \frac{z}{s}$$

Three of the values  $x_i'$ ,  $y_i'$  and  $z_i'$ , can be reduced to zero by a suitable choice of the system of coordinates.

If the dimensions and stresses of two struts, for example,

$$\alpha^{1,0}, \alpha^{2,0} \quad \text{and hence} \quad \frac{1}{e^{1,0}}, \frac{1}{e^{2,0}}$$

are known, the value  $1/e^{3,0}$  of the third strut, at which the system would buckle, can then be calculated. If the value  $1/e^{3,0}$  of the last strut (with attention to the sign) is larger, the dimensions then suffice. The calculation is considerably simplified when the three are perpendicular to one another (Fig. 5). The buckling condition then becomes

$$\left( \frac{1}{e^{1,0}} + \frac{1}{e^{2,0}} \right) \left( \frac{1}{e^{1,0}} + \frac{1}{e^{3,0}} \right) \left( \frac{1}{e^{2,0}} + \frac{1}{e^{3,0}} \right) = 0.$$

In this case, buckling occurs when the buckling condition of the most highly stressed component  $s_1 s_2$ ,  $s_1 s_3$  or  $s_2 s_3$  is fulfilled for buckling in its plane.

c) Closed rectangular frame as a system of struts

The last-mentioned case under b comes in question when a flange member and two adjoining uprights, belonging to a cross wall perpendicular to the side walls, are combined in one group (Fig. 6).

Instead of assuming flexible joints at 2 and 4, or 1 and 3, for estimating the buckling load in the uprights, it is advisable, as regards buckling with respect to the planes of the longitudinal walls, to regard the uprights as belonging to the group of bars of the transverse structure.

In the buckling of the transverse frame in its plane, the joint "i" turns ( $i = 1, 2, 3, 4$ ). Consequently

$$\begin{aligned} M_i^{i-1, i} & \text{ acts on } i-1, i \text{ at "i"} \\ M_i^{i, i+1} & \text{ " " } i, i+1 \text{ " "i"} \end{aligned}$$

The cross section turns

$$\begin{aligned} \text{"i" on } i-1, i & \text{ by } M_i^{i-1, i} e^{i-1, i} + M_{i-1}^{i-1, i} \dot{e}^{i-1, i} \\ \text{"i" " } i, i+1 & \text{ " } M_i^{i, i+1} e^{i, i+1} + M_{i+1}^{i, i+1} \dot{e}^{i, i+1} \end{aligned}$$

$e$  = unit rotation of a strut end produced by a moment on the same end  $M = 1$  (see 4a,  $\alpha$ ).  $\dot{e}$  = unit rotation of one strut

end, due to a moment on the other end  $M = 1$  in the direction of  $M = 1$  (See below).

For equilibrium

$$M_i^{i-1,i} + M_i^{i,i+1} = 0 \quad (i = 1, 2, 3, 4, 1) \quad (1)$$

For continuity

$$\left. \begin{aligned} M_i^{i-1,i} e^{i-1,i} + M_{i-1}^{i-1,i} \dot{e}^{i-1,i} &= \\ M_i^{i,i+1} e^{i,i+1} + M_{i+1}^{i,i+1} \dot{e}^{i,i+1} & \end{aligned} \right\} \quad (2)$$

$$(i = 1, 2, 3, 4, 1)$$

These equations enable finite values of  $M$  and consequently finite rotations of the joints, when

$$\begin{vmatrix} e^{4 \cdot 1} + e^{1 \cdot 2} & - \dot{e}^{1 \cdot 2} & 0 & - \dot{e}^{4 \cdot 1} \\ - \dot{e}^{1 \cdot 2} & e^{1 \cdot 2} + e^{2 \cdot 3} & - \dot{e}^{2 \cdot 3} & 0 \\ 0 & - \dot{e}^{2 \cdot 3} & e^{2 \cdot 3} + e^{3 \cdot 4} & - \dot{e}^{3 \cdot 4} \\ - \dot{e}^{4 \cdot 1} & 0 & - \dot{e}^{3 \cdot 4} & e^{3 \cdot 4} + e^{4 \cdot 1} \end{vmatrix} = 0 \quad \left( \begin{array}{l} \text{Buckling} \\ \text{condition} \end{array} \right)$$

The unit rotation  $\dot{e}$  (Cf. H. Muller-Breslau, Reference 9) is

$$\dot{e} = - \frac{s}{E J} \frac{1}{\alpha^2} \left( \frac{\alpha}{\sin \alpha} - 1 \right)$$

whereby

$$\alpha = s \sqrt{\frac{S}{E J}} + S = \text{compression}$$

or 
$$\dot{e} = - \frac{s}{E J} \frac{1}{6} \quad \text{provided } S = 0$$

or

$$\dot{e} = - \frac{s}{E J} \frac{1}{\alpha^2} \left( 1 - \frac{\alpha}{\sin \alpha} \right)$$

whereby

$$\alpha = s \sqrt{\frac{S}{E J}}; \quad + S = \text{tension.}$$

The buckling condition is considerably simplified, when the upper cross beam is dimensioned and stressed like the lower one and the left upright is dimensioned and stressed like the right one. It then reads

$$(e^V + e^R - \dot{e}^V + \dot{e}^R) (e^V + e^R + \dot{e}^V - \dot{e}^R)$$

$$(e^V + e^R + \dot{e}^V + \dot{e}^R) (e^V + e^R - \dot{e}^V - \dot{e}^R) = 0$$

$$V = \text{upright.} \quad R = \text{cross beam.}$$

Each bracket, put equal to zero, furnishes a buckling condition.

At

$$e^V + e^R - \dot{e}^V - \dot{e}^R = 0$$

all the members buckle in the form of a bow.

If the left upright is dimensioned and stressed like the right one, we obtain

$$[e^V + e^{Ro} + \dot{e}^{Ro}] [e^V + e^{Ru} + \dot{e}^{Ru}] = (\dot{e}^V)^2,$$

~~whereby the upper signs correspond to the bow-shaped buckling.~~

## E x a m p l e 2

	<u>Upright</u> (1.2)=(3.4)	<u>Cross beam</u> (2.3) (4.1)	
s (cm)	100	80	80
$J_s$ (cm <sup>4</sup> )	0.3647	0.1513	0.030
F (cm <sup>2</sup> )	0.660	0.406	0.181
S (kg) (load)	780	400	0
E (kg/cm <sup>2</sup> )	$2 \times 10^6$	$2 \times 10^6$	$2 \times 10^6$
Natural buckling load (kg)	715	463	93
$\alpha = s \sqrt{\frac{S}{E J}}$ (Fig. 2)		2.91	0
$\frac{1}{\alpha^2} \left(1 - \frac{\alpha}{\tan \alpha}\right)$		1.5	0.33
$\frac{1}{\alpha^2} \left(\frac{\alpha}{\sin \alpha} - 1\right)$		1.3	0.17
Unit rotation $e \left(\frac{1}{\text{kg cm}}\right)$		$4 \times 10^{-4}$	$4.5 \times 10^{-4}$
Unit rotation $\ddot{e} \left(\frac{1}{\text{kg cm}}\right)$		$-3.5 \times 10^{-4}$	$-2.3 \times 10^{-4}$

Buckling condition:

$$[e^{1.2} + (e^{2.3} - \dot{e}^{2.3})] [e^{1.2} + (e^{4.1} - \dot{e}^{4.1})] = (\dot{e}^{1.2})^2$$

$$[e^{1.2} + 7.5 \times 10^{-4}] [e^{1.2} + 6.8 \times 10^{-4}] = (\dot{e}^{1.2})^2$$

$$e^{1.2} = 1.4 \times 10^{-4} \frac{1}{\alpha^2} \left(1 - \frac{\alpha}{\tan \alpha}\right); \dot{e}^{1.2} = -1.4 \times 10^{-4} \frac{1}{\alpha^2} \left(\frac{\alpha}{\sin \alpha} - 1\right).$$

It is fulfilled by

$$\alpha_{1.2} = 3.261.$$

This value is reached when

$$S_{1.2} = \frac{E J \alpha^2}{s^2} = 770 \text{ kg}$$

acts in the uprights.

Since  $S_{1.2} = 780 \text{ kg}$ , the frame is not safe against buckling. With  $J'_{1.2} = 0.3755 \text{ cm}^4$  the buckling condition gives  $\alpha_{1.2} = 3.256$ . This is reached at

$$S_{1.2} = \frac{E J' \alpha^2}{s^2} = 792 \text{ kg}$$

With  $J'$  the frame is therefore safe against buckling in its plane. The stresses in the individual members lie below the elastic limit. The calculation with

$$E = 2 \times 10^6 \text{ kg./cm}^2$$

is therefore justified.

#### d) Generalization

If, due to  $s_2$  alone, the end-fixation effect does not suffice for the requisite fixation of a strut  $s_1$  against buckling in the plane  $s_1/s_2$ , another strut  $s_3$  can be added, which lies in the plane  $s_1/s_2$  and is combined with  $s_1$  and  $s_2$  in one joint.  $s_2/s_3$  then represents a strut with the fixation coefficient (See Section a)

$$\frac{1}{e^{2.3}} = \frac{1}{e^{1.2}} + \frac{1}{e^{1.3}}$$

As regards the combination of several struts in groups, see the references.

### 5. Unelastic Buckling

The stresses  $\sigma$  in a group of struts in the buckling state (in which there may also be tensile stresses) lie mostly above the elastic limit. It must then be remembered that, in the case of such stresses, reduced values of  $E'$  hold good for the modulus of elasticity (See References 1 and 10).

For the estimation of the moduli of elasticity  $E'$  corresponding to the existing compressive stresses, the customary buckling formulas of the unelastic zone can be used.

If such a formula (Tetmajer, Natalis, etc.) reads

$$\sigma_k = f(\lambda) \quad (\lambda = \text{slenderness ratio}),$$

and if

$$\sigma_k = \frac{E' J \pi^2}{l^2 F} = E' \frac{\pi^2}{\lambda^2}$$

in the unelastic zone according to a suggestion of Engesser, the elimination of  $\lambda$  then yields

$$E' = F(\sigma_k).$$

From Tetmajer's formula  $\sigma_k = a + b \lambda$  follows

$$E' = \frac{\sigma_k (\sigma_k - a)^2}{b^2 \pi^2}$$

If there are in the groups tension members whose elastic limit is exceeded, suitable allowance can be made for the reduction of the modulus of elasticity by regarding them as free from

stress, so that

$$e = \frac{1}{3} \frac{s}{E J}$$

or

$$\dot{e} = \frac{1}{6} \frac{s}{E J}$$

If the yield limit is exceeded in a tension member of the group, it is advisable to disregard the stiffness of this member until an experimental solution of this problem is obtained.

If one desires to calculate the load factors at which buckling develops in a group, he first uses the original value  $E$ , determines the stresses corresponding to the buckling condition and then calculates the corresponding  $E'$  values of the individual members. For this purpose he determines anew the buckling condition and the corresponding stresses in the members. If the  $E''$  values differ much from the  $E'$  values, the process, under some conditions, must be repeated several times. (For increased clearness, Bleich here introduces reduction factors of the inertia moment.)

### E x a m p l e 3

	Strut 1.0	Strut 2.0
s (cm)	75	75
J (cm <sup>4</sup> )	0.3647	0.3647
F (cm <sup>2</sup> )	0.660	0.660
S (kg) (load)	1550	800
E (kg/cm <sup>2</sup> )	$2 \times 10^6$	$2 \times 10^6$

Tetmajer formula	$\sigma_k = 3400 - 12.7 \lambda$	
Stress (kg/cm <sup>2</sup> )	2350	1210
Zone	Unelastic	Elastic
$E' \text{ (kg/cm}^2\text{)}$	$1.635 \times 10^8$	$2 \times 10^6$
$\alpha = s \sqrt{\frac{S}{E'J}}$	3.84	2.48
$\frac{1}{\alpha^2} \left(1 - \frac{\alpha}{\tan \alpha}\right)$ (Fig. 2)	-0.25	0.68
Unit rotation (1 kg cm)	$-0.315 \times 10^{-4}$	$0.7 \times 10^{-4}$
Buckling condition	$e^{1.0} + e^{2.0} = 0$	
For	$e^{1.0} = -0.315 \times 10^{-4}$ it is fulfilled, when	
	$e^{2.0} = +0.315 \times 10^{-4}$ .	

The obtained value of  $e^{2.0}$  is greater, and the system is therefore not safe against buckling. (If the calculation had been wrongly made with  $E = 2 \times 10^6$  kg/cm<sup>2</sup> in both members, it would have shown safety against buckling.)

## 6. End-Fixation Effects in the Construction of Bridges and High Buildings

According to "Methods of Calculation for the Design of Iron Railway Bridges of the German Railway System" (Reference 11) and according to the ministerial decree II, 9, 156 regarding the specifications on safe stresses in ingot steel, etc., (Reference 12), no fixation effect can be taken into account in bridges or high structures in the case of flange members and

the end diagonals of trapezoidal girders. In general, according to these specifications, the dimensions of the web members, used to prevent buckling from the plane of the trusses, are determined without consideration of any fixation effect.

In dimensioning the web members so as to prevent buckling in the plane of the truss, allowance can be made for any end-fixation effect that can be calculated with the distance between the centers of gravity of the end groups of rivets instead of the length of the frame. Moreover, in bridge construction, in the design of uprights which form a bending-resistant framework with the corresponding cross members, the distance between the centers of the rigid joints may be included in the calculation instead of the length of the frame.

It should be noted that the bending stress generally determines the dimensions of the cross beams and that the latter are not therefore to be regarded as members subjected to normal stresses. In airplane construction, however, the cross members can be fully utilized against buckling, i.e., they can be stressed with their "natural" buckling load, so that they will have no end-fixation effect.

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## Concluding Remarks

The end-fixation effect enters into the problem, when a compression member in a framework is rigidly joined to adjacent members which are not fully used in tension, or are unstressed, or are not fully utilized in compression (i.e., are stressed below their natural buckling load). These oppose the buckling of the member under consideration.

Any strut  $s_1$ , which is already used for the fixation of another strut, does not generally come into the question of the fixation of yet another strut, unless more accurate methods of calculation are employed because, if the system  $s_1/s_2$  is sufficiently utilized in compression, it offers hardly any resistance to buckling caused by another strut.

Hence, if we assemble (according to Bleich's suggestion) each pair of flange members  $s_1/s_2$  or  $s_3/s_4$  stressed in tension for the fixation of a cross beam  $V_1$  or  $V_3$  against buckling in the given plane, it will be advisable to give the strut  $V_2$  such a form that, of itself, it will be safe against buckling (Fig. 7). The conditions are different in bridges since, due to the changing position of the load, two adjacent web members are not generally subjected simultaneously to the maximum stress.

In practice it will be worth while to consider the end-fixation effect in uprights which, on the one hand, resist buckling in the plane of the truss on the tension flange and, unless

there are brace wires, are attached to the rigid tension members and, on the other hand, belong to transverse frames perpendicular to the plane of the truss, whose members are stressed below their "natural" buckling load. Moreover, it will be worth while in the case of a compression flange, which is followed immediately by a parallel compression member stressed below its "natural" buckling load. The latter case occurs when, in the usual manner, the same tubular cross section is retained in several successive panels.

If there are struts which resist bending, they may be regarded as pinned to the tension flange and tension uprights against buckling in the plane of the truss. It is natural to combine them, with respect to buckling out of the plane of the truss, with the adjoining uprights of the neighboring trusses into oblique frames. In doing this, care must be exercised that the members meeting in common joints shall not meet at right angles.

One is expressly warned against the unlimited application of the above methods to the estimation of the end-fixation effect, with reference to the statements in Section 1 (page 1). Especially must it be borne in mind that almost all members are ~~distorted in welding.~~ On account of the effect of the secondary stresses in the vicinity of the annealed zones, it is advisable to calculate as though the members near these zones were flexibly connected with the rest of the framework. Then the calcula-

tion of the end-fixation effect would only have to show whether the last assumption is permissible in each case.

Of necessity the calculations give a large number of buckling cases, since the trigonometrical functions are periodic. We are naturally concerned only with the minimum buckling case. In the case of a strut, which is "fixed" by neighboring struts not fully utilized, this is generally more than the simple and less than the double natural buckling load.

The main purpose of this communication is to give references to the literature on the subject and also to furnish the constructor with a general survey of the simplest methods for estimating the end-fixation effect. The details are to be found in the documents referred to.

A few examples and nomograms will be published in a later number of Zeitschrift für Flugtechnik und Motorluftschiffahrt. A thorough investigation of the end-fixation effect has been begun by the D.V.L.

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~~Translation by Dwight M. Miner,~~  
National Advisory Committee  
for Aeronautics.

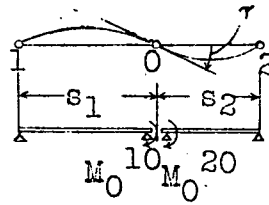
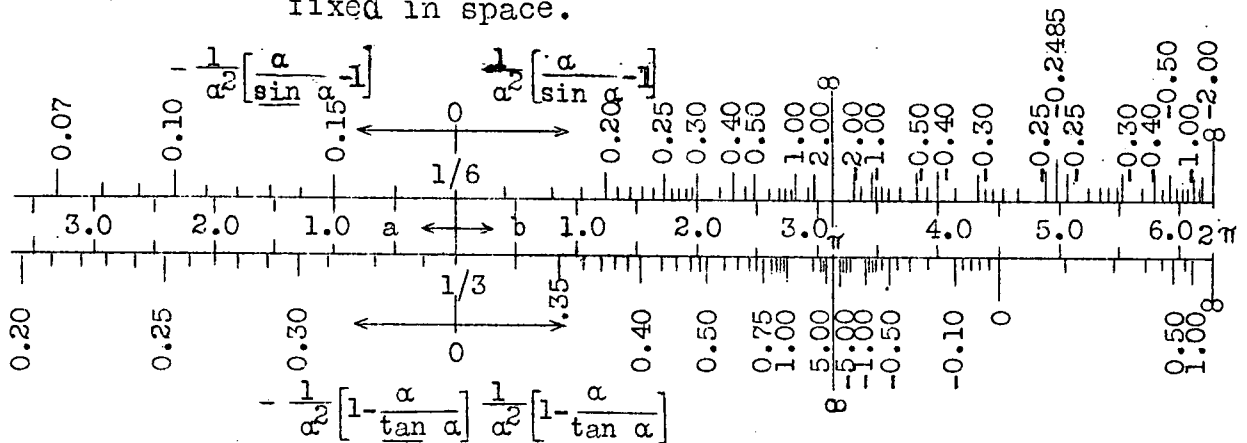


Fig.1 Buckling of a group of two successive struts in the same direction, with junction points fixed in space.



$a = \alpha$  [Tension member]

$b = \alpha$  [Compression member]

Fig.2 Representation of the expressions  $\frac{1}{\alpha^2} (1 - \frac{\alpha}{\tan \alpha})$ ,  $\frac{1}{\alpha^2} (\frac{\alpha}{\sin \alpha} - 1)$  etc.

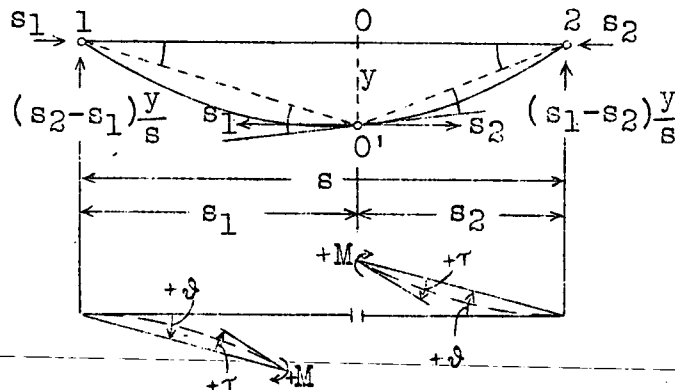


Fig.3 Buckling of a group of two successive struts in the same direction, with middle joint not fixed in space.

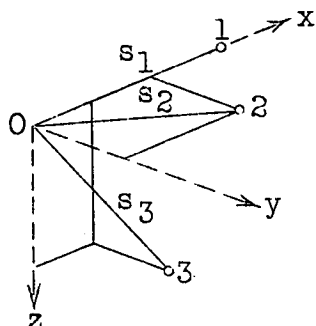


Fig. 4 Group of three spatially-related struts.

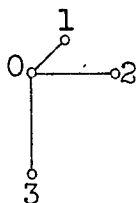


Fig. 5 Group of three struts perpendicular to one another.

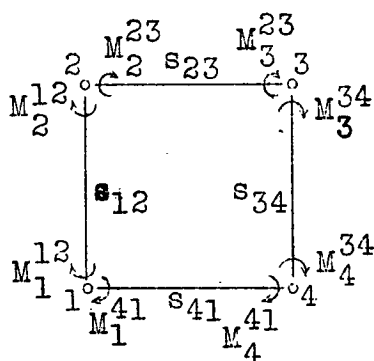


Fig. 6 Buckling of a frame in its plane.

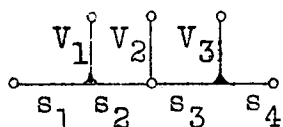


Fig. 7 End-fixation effect of a tension flange on the web members.